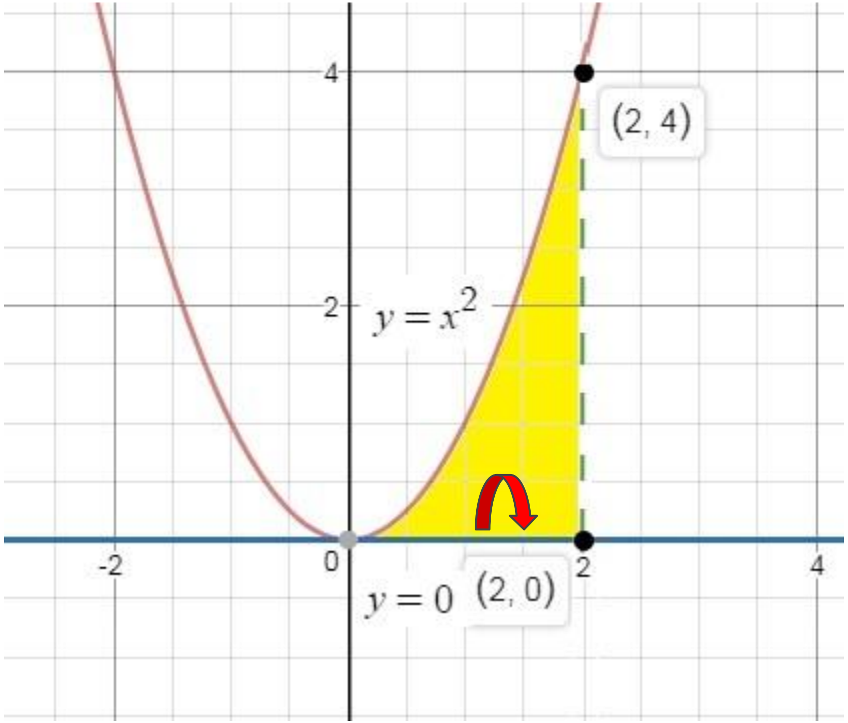
Applications of Definite Integrals

Finding the Volume of the Solid of Revolution Using Disc Method

Volume of the Solid of Revolution

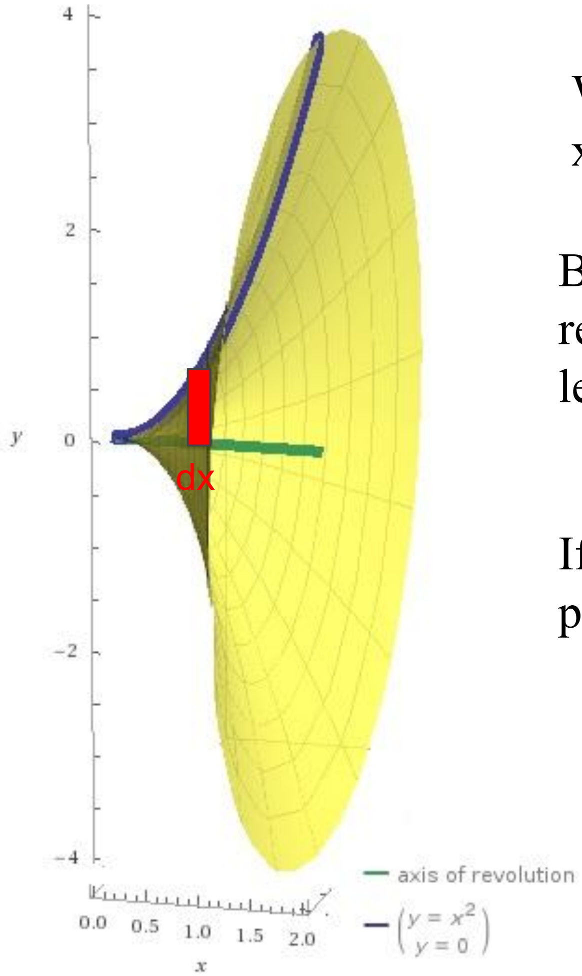
- Disc Method Around the X-axis
- Disc Method Around the Y-axis
- Disc Method Around the Horizontal Line
- Disc Method Around the Vertical Line

Disc Method Around The X-axis



We have a region enclosed by the graph of $y = x^2$ and y = 0 from x = 0 to x = 2.

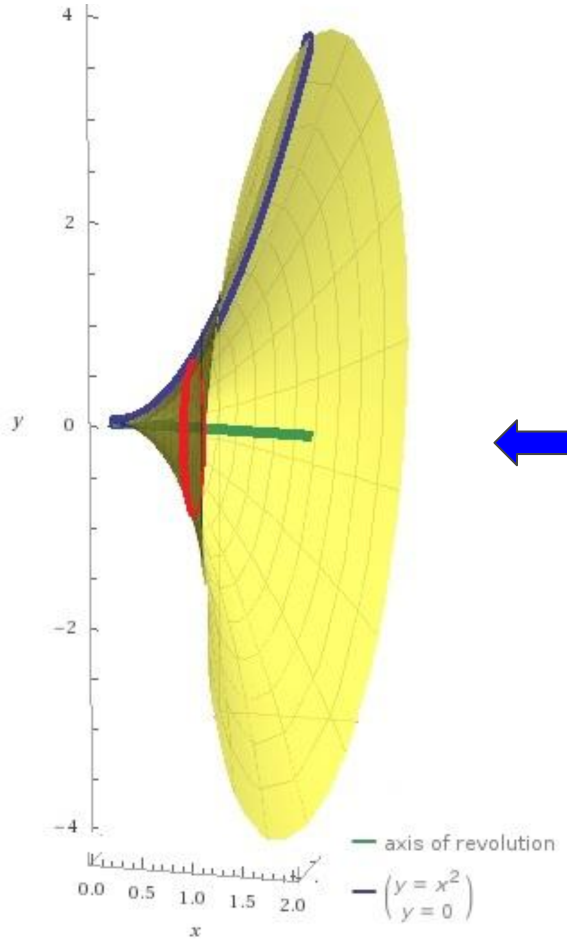
This region will be rotated around the x-axis.

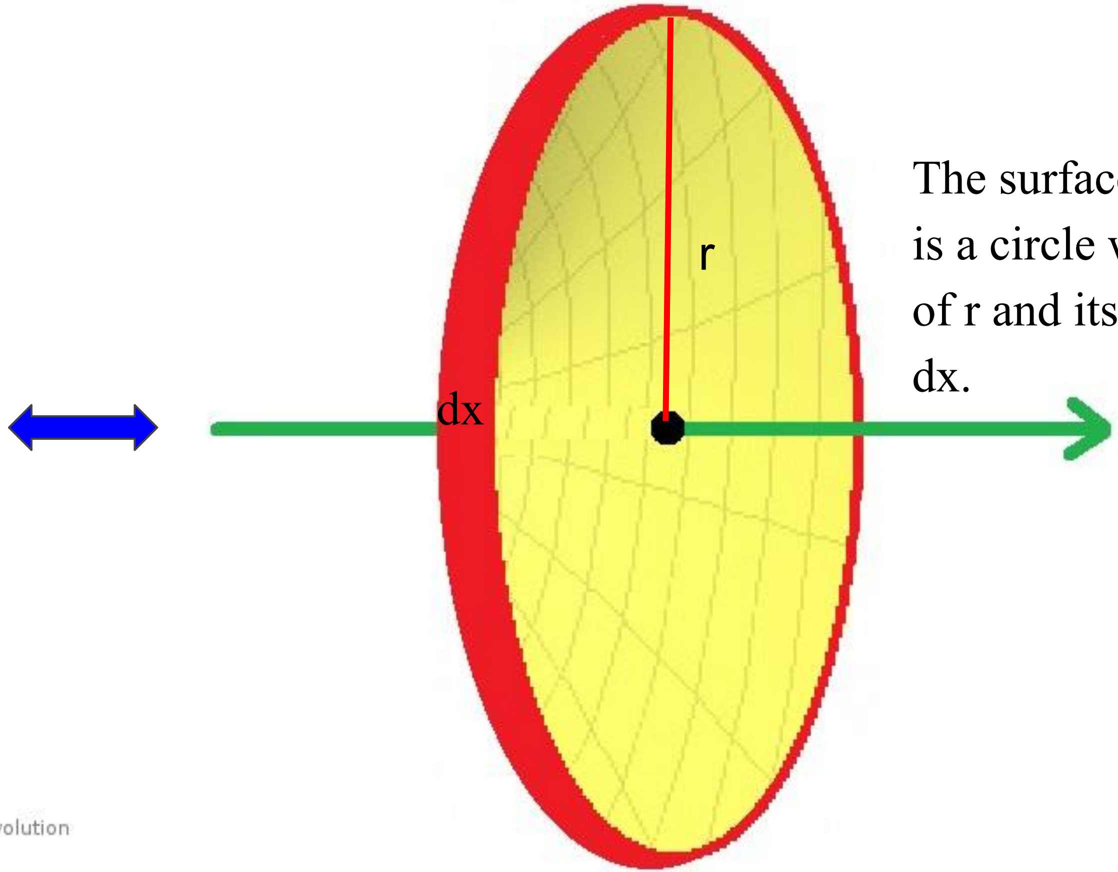


When the enclosed region is rotated around the x-axis, the resulting solid has no holes or gaps.

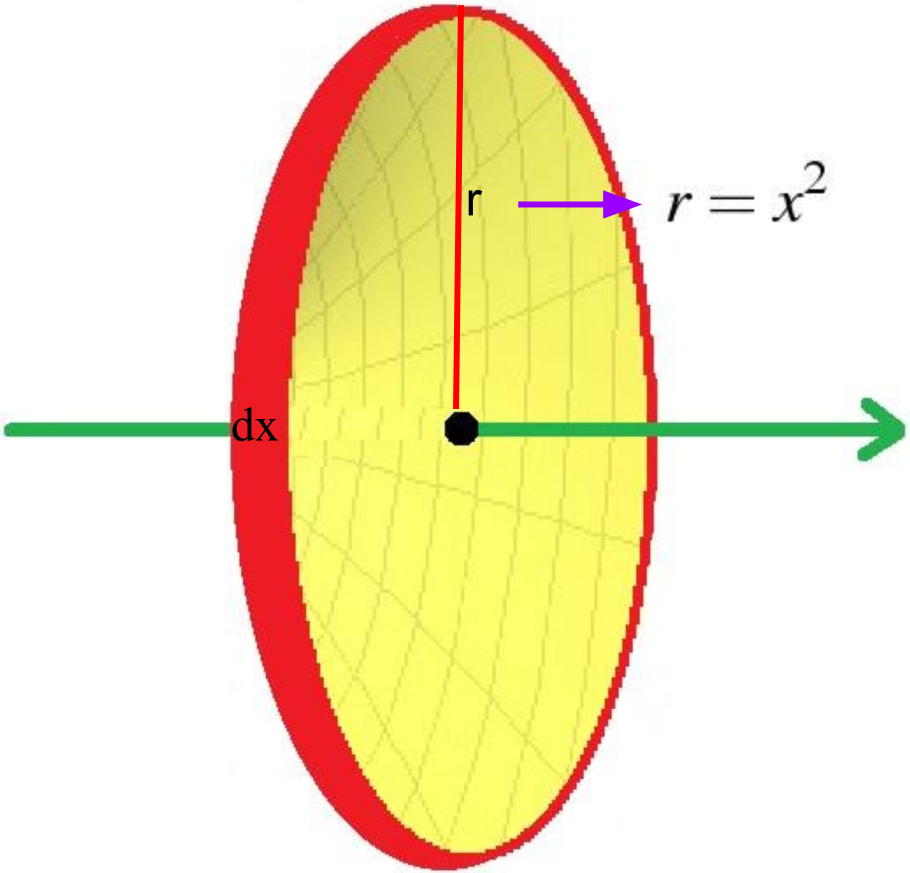
By Riemann Sum Theorem, we have many infinite rectangles and each rectangle has a width of dx and length of $y = x^2$.

If we will rotate a rectangle, we will have a disc perpendicular to the axis of revolution.





The surface of the disc is a circle with a radius of r and its thickness is dx.



Volume Formula of a Cylinder

$$Volume = \pi r^2 h$$

Area Formula of a Circle

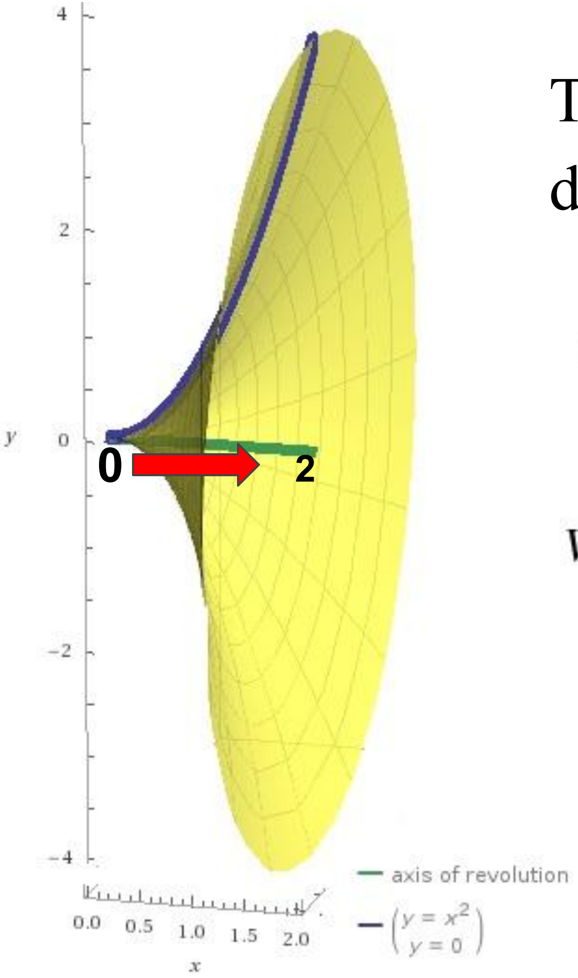
$$A(x) = \pi r^2$$

$$A(x) = \pi \cdot (x^2)^2$$

$$A(x) = \pi \cdot x^4$$

The Volume of the Disc

$$V(x) = \pi \cdot x^4 dx$$



The volume of the entire solid using definite integral from x = 0 to x = 2 is:

$$V = \int_0^2 \pi \cdot x^4 dx$$

$$V(x) = \pi \int_0^2 x^4 dx$$

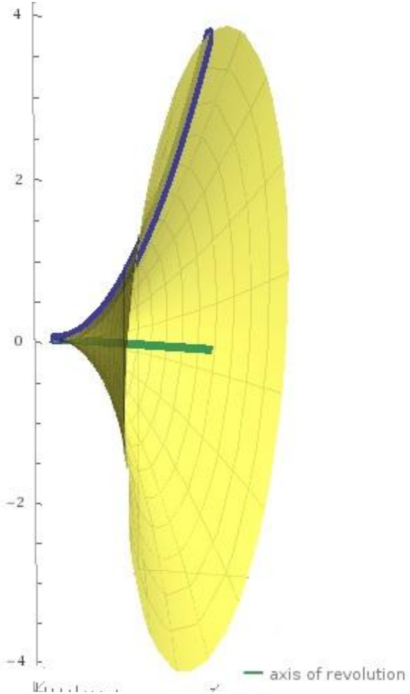
We evaluate the integral.

$$V = \pi \int_0^2 x^4 dx$$

$$V = \pi[\frac{x^5}{5}]_0^2$$

$$V = \pi \cdot [\frac{2^5}{5} - \frac{0^5}{5}]$$

$$V = \frac{32\pi}{5}$$



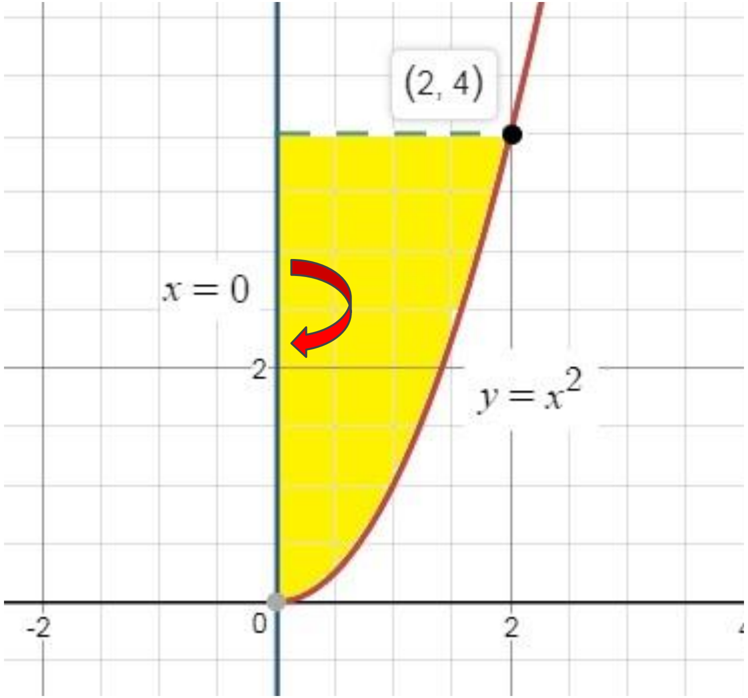
Therefore, the volume of the solid is $\frac{32\pi}{5}$ *cubic units.*

The Volume Formula of the Solid of Revolution Around X-axis Using Disc Method

$$V(x) = \pi \int_a^b (f(x))^2 dx$$

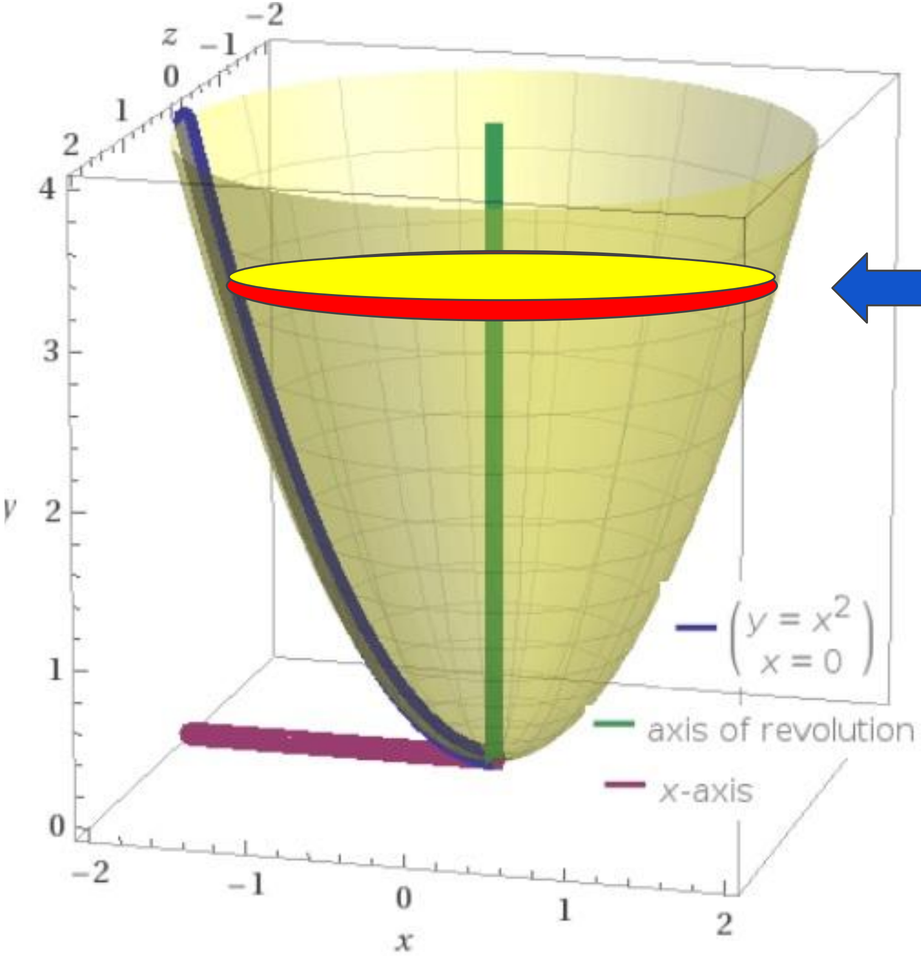
Where,
f(x) is the function and the boundaries for x is between a and b.

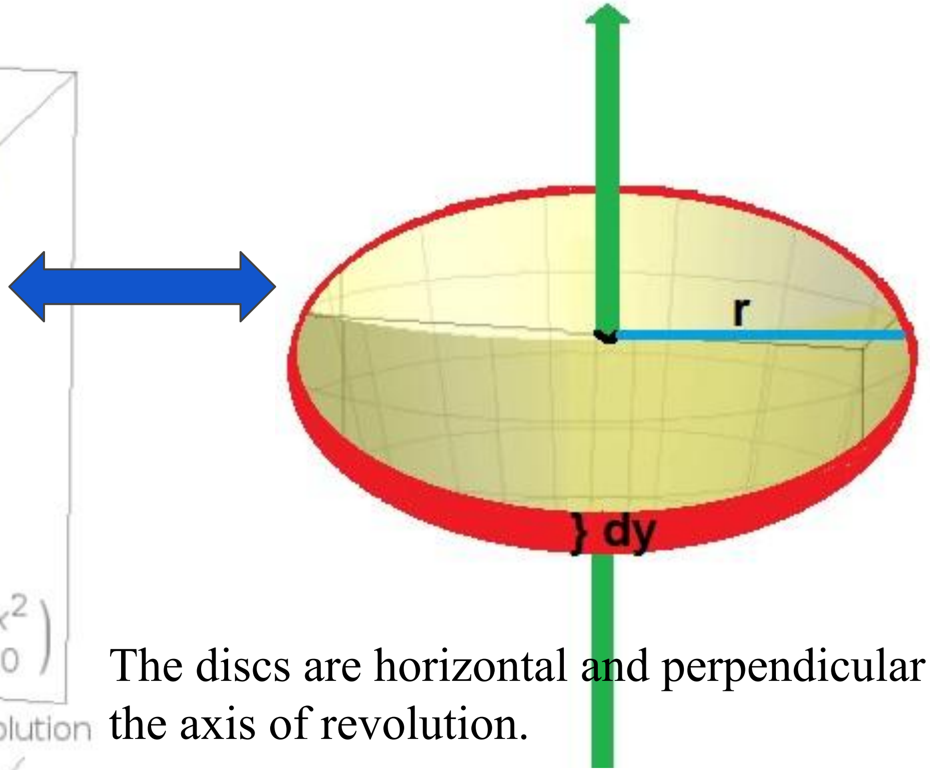
Disc Method Around the Y-axis



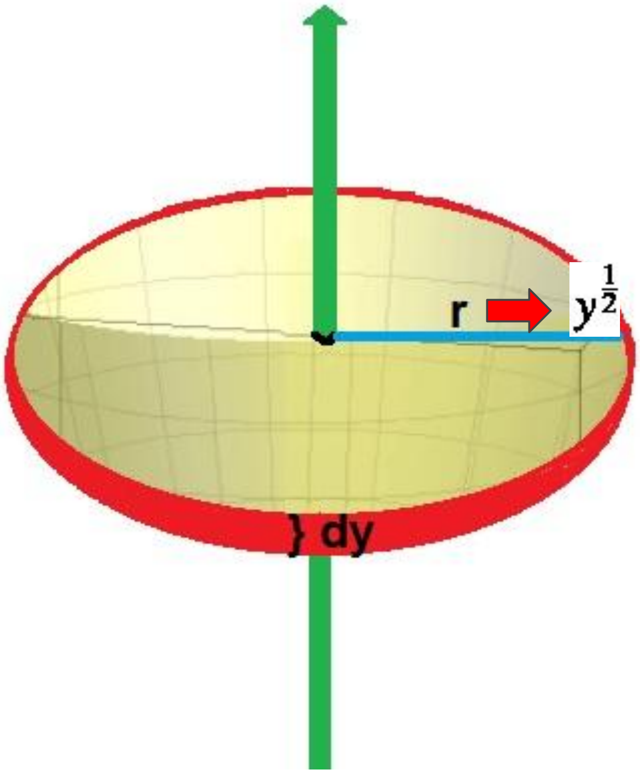
We have a region enclosed by the graph of $y = x^2$ and x = 0 from y = 0 to y = 4.

This region will be rotated around the y-axis.





The discs are horizontal and perpendicular to the axis of revolution.



The radius of each disc is r = x, and its thickness is dy.

We should find all of our components in terms of y.

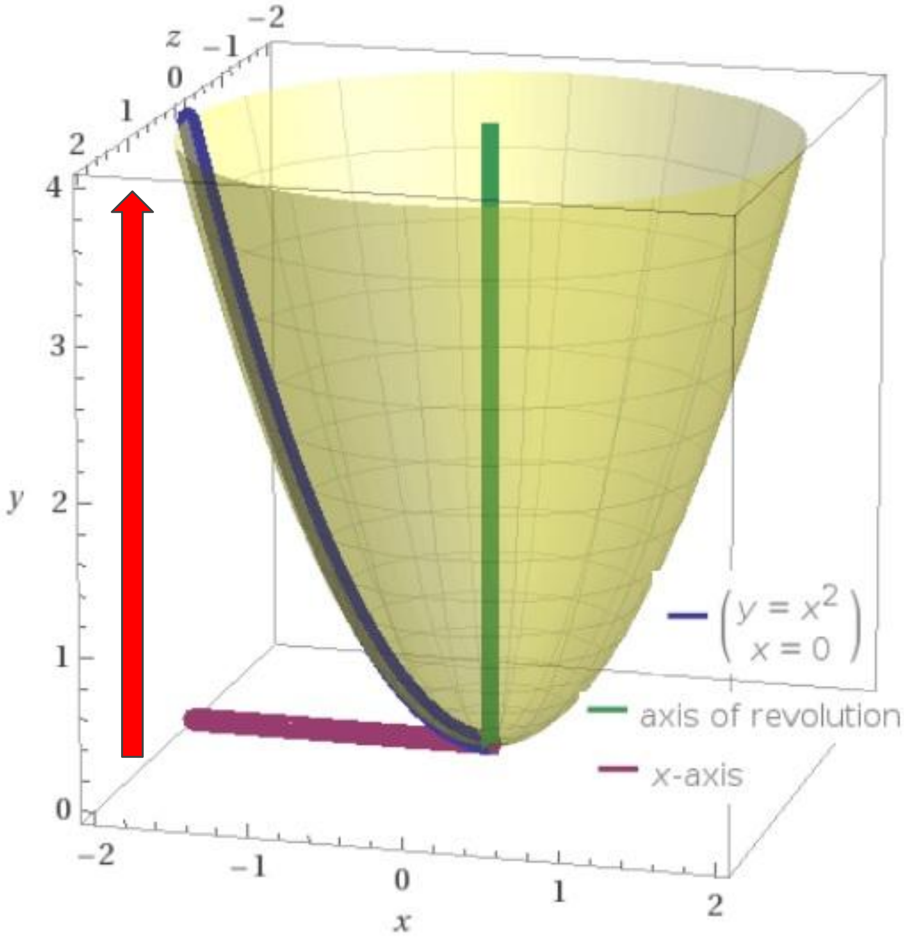
$$y = x^2 \rightarrow x = y^{\frac{1}{2}}$$

So, it follows that the area of a face of one disc is

$$A = \pi r^2$$
$$A(y) = \pi (y^{\frac{1}{2}})^2$$
$$A(y) = \pi y$$

The volume of one disc is

$$V(y) = \pi y dy$$



We have to sum up the the volumes of infinitely many discs between y = 0 to y = 4.

The volume of the entire solid is given by

$$V(y) = \int_0^4 \pi y dy$$

$$V(y) = \pi \int_0^4 y dy$$

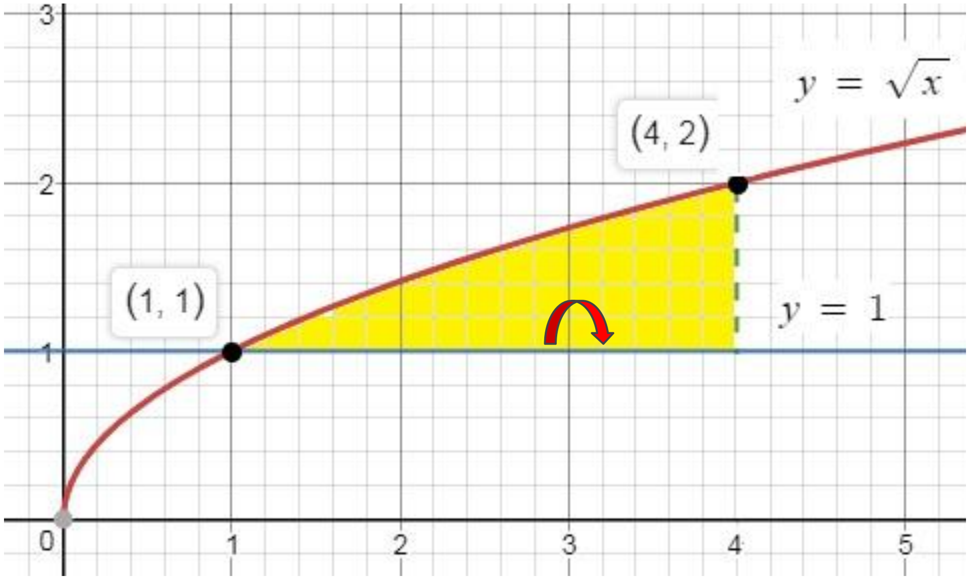
The Volume Formula of the Solid of Revolution Around Y-axis Using Disc Method

$$V(y) = \pi \int_{c}^{d} (f(y))^2 dy$$

Where,

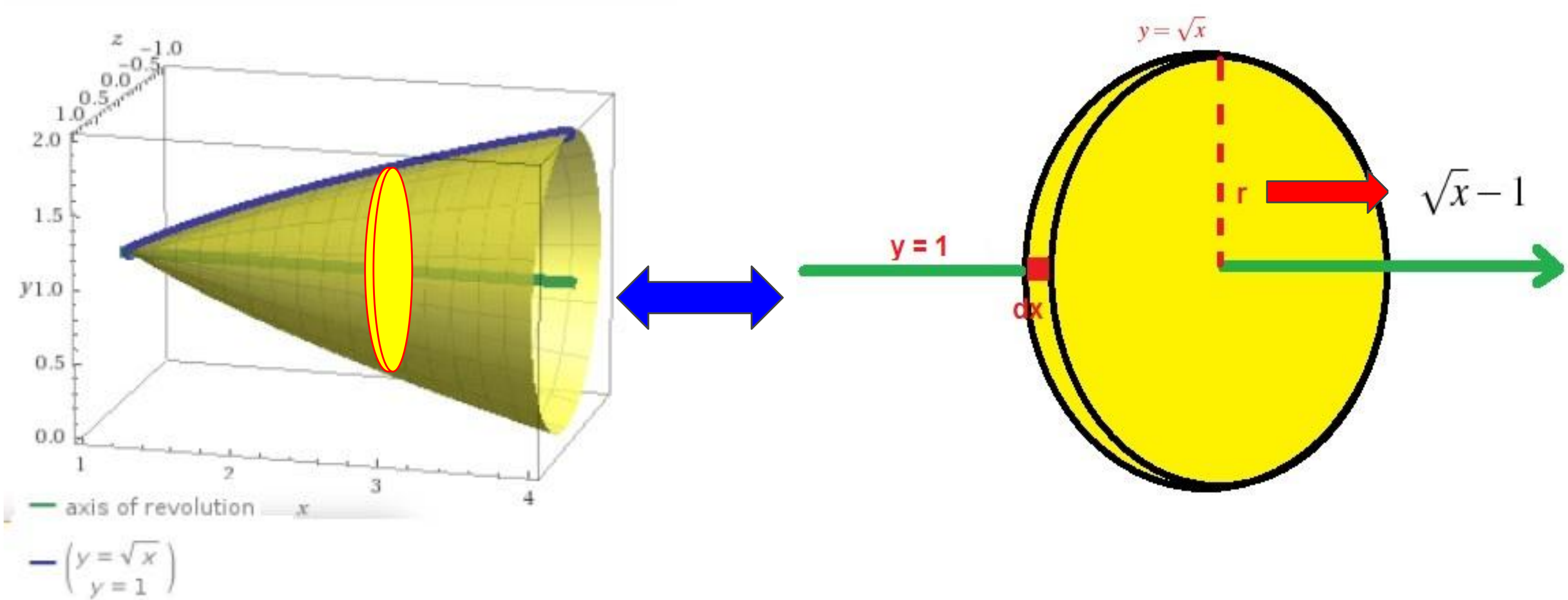
f(y) is the function in terms of y and the boundaries for y is between c and d.

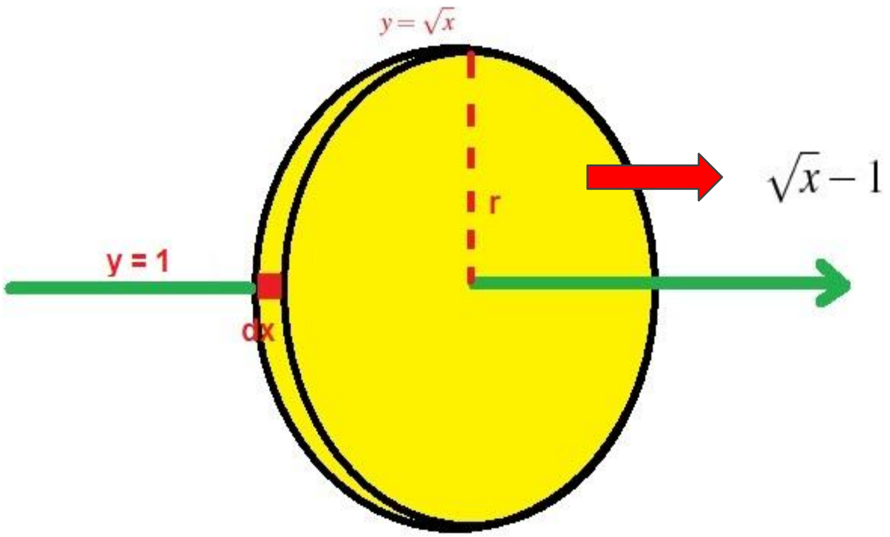
Disc Method Around the Horizontal Line



We have here a region enclosed by the graph of $y = \sqrt{x}$ and $y = 1$ from $x = 1$ to $x = 4$.

This region will be rotated around a horizontal line $y = 1$.


z
y
x
axis of revolution
$y = \sqrt{x}$
$y = 1$
$y = \sqrt{x}$
r
$\sqrt{x} - 1$
y = 1
dx

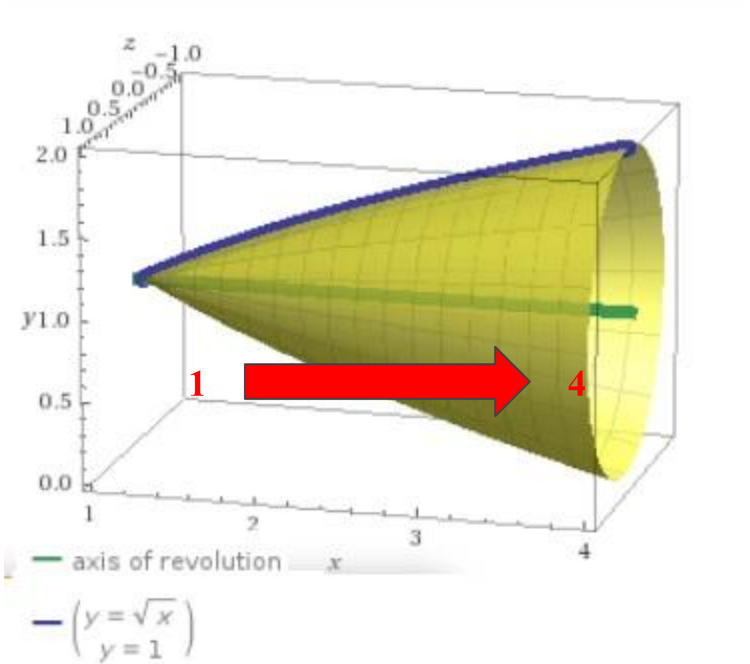


So, the area of the face of each disk is

$$A = \pi r^2$$
$$A(x) = \pi(\sqrt{x}-1)^2$$

The volume of each disk is

$$V(x) = \pi(\sqrt{x}-1)^2 dx$$



We want to sum up the volumes of infinitely many of these discs between x = 1 to x = 4.

The volume of the entire solid is given by

$$V(x) = \int_1^4 \pi(\sqrt{x} - 1)^2 dx$$

$$V(x) = \pi \int_1^4 (\sqrt{x} - 1)^2 dx$$

The Volume Formula of the Solid of Revolution Around a Horizontal Line Using Disc Method

$$V(x) = \pi \int_a^b (f(x) - g(x))^2 dx$$

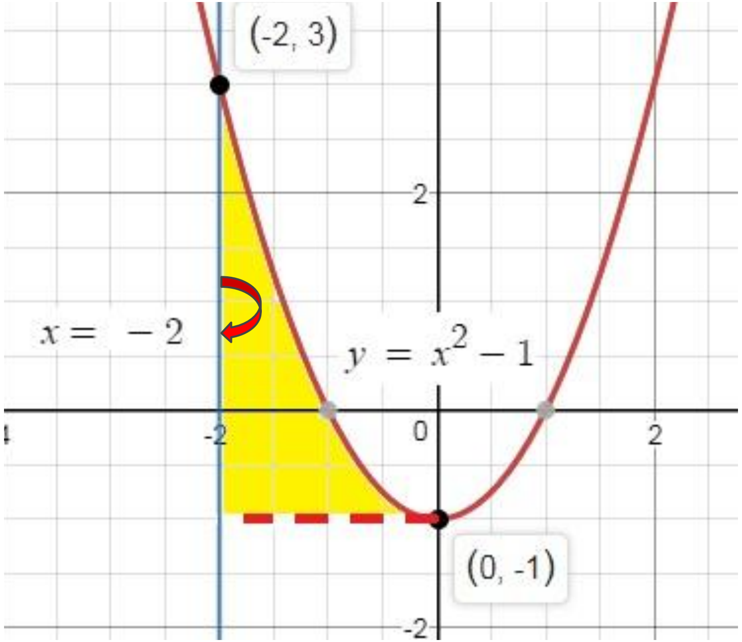
Where,

The boundaries of x is between a and b

f(x) is the curve function; and

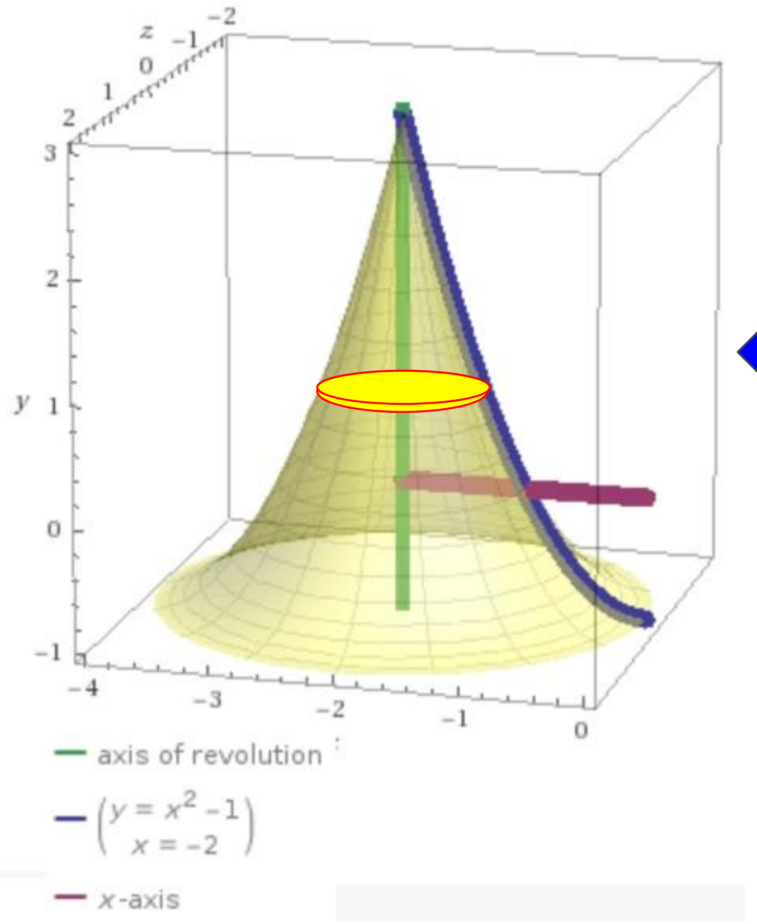
g(x) is the horizontal line.

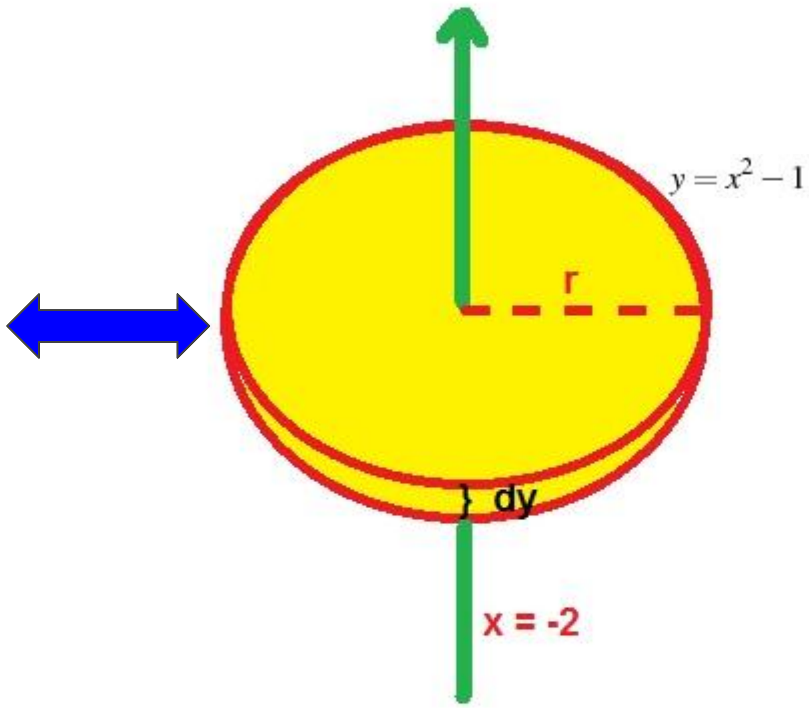
Disc Method Around the Vertical Line

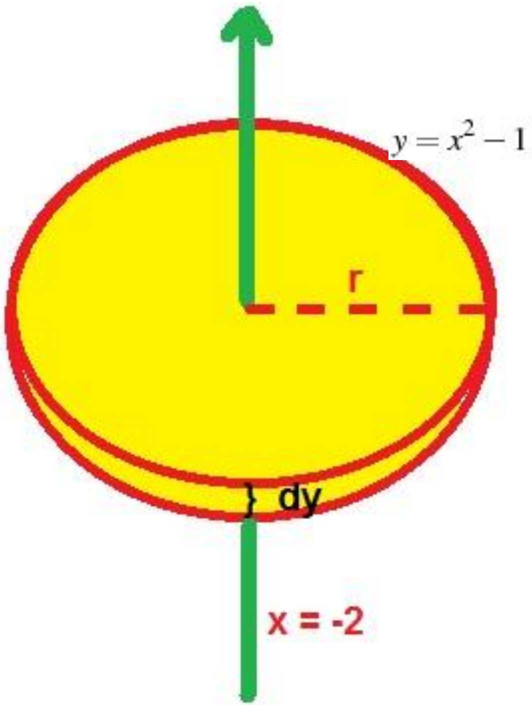


We have a region enclosed by the graph of $y = x^2 - 1$ and x = -2 from y = -1 to y = 3.

This region will be rotated around the vertical line x = -2.


z
-2
-1
0
1
2
3
2
1
0
-1
y
-4
-3
-2
-1
0
axis of revolution
$y = x^2 - 1$, $x = -2$
x-axis


$y = x^2 - 1$
r
dy
x = -2



The discs are horizontal and so the radius of each disc is r = x and its thickness is dy.

With this, we should find all of our components in terms of y.

$$y = x^2 - 1 \rightarrow x = \sqrt{y+1}$$

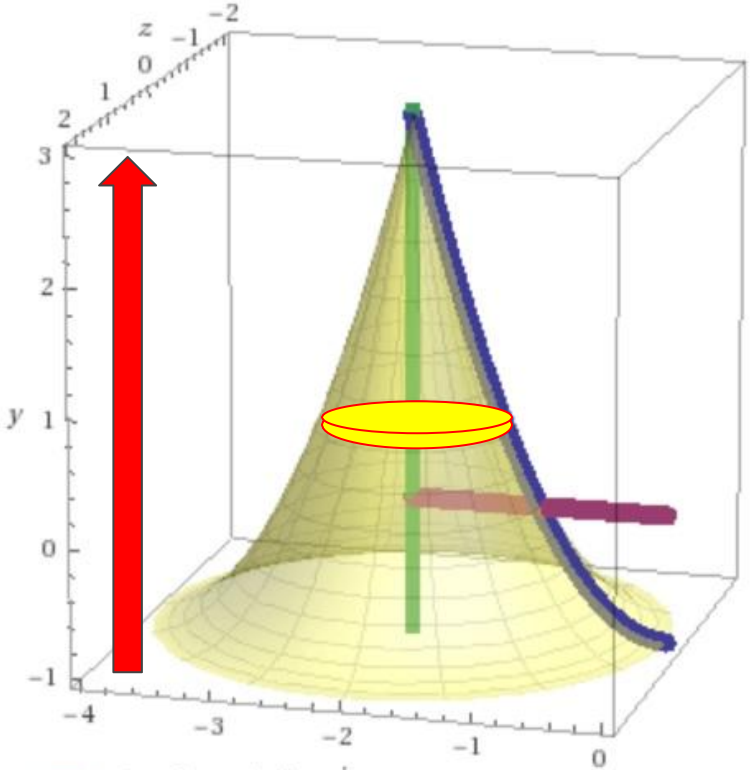
The radius of each disc is

$$r = \sqrt{y+1} - (-2)$$

$$r = \sqrt{y+1} + 2$$

The area of each disc is

$$A(y) = \pi(\sqrt{y+1} + 2)^2$$



The volume of one disc is

$$V(y) = \pi(\sqrt{y+1}+2)^2 dy$$

We have to sum up the volumes of infinitely many of these discs between y = -1 to y = 3.

The volume of the entire solid is given by

$$V(y) = \int_{-1}^{3} \pi(\sqrt{y+1}+2)^2 dy$$

$$V(y) = \pi \int_{-1}^{3} (\sqrt{y+1}+2)^2 dy$$

The Volume Formula of the Solid of Revolution Around a Vertical Line Using Disc Method

$$V(y) = \pi \int_{c}^{d} (f(y) - g(y))^2 dy$$

Where,

The boundaries of y is between c and d

f(y) is the curve function in terms of y; and

g(y) is the vertical line.

Summary

- In this method, the axis of revolution is the boundary of the plane region and the cross sections are taken perpendicular to the axis of revolution.

- The cross-section of a disc is a circle with $A = \pi r^2$, the volume of each disc is its area times its thickness/depth (dx/dy).

Formulas

- Disc Method Around the X-axis : $V(x) = \pi \int_a^b (f(x))^2 dx$
- Disc Method Around the Y-axis: $V(y) = \pi \int_c^d (f(y))^2 dy$
- Disc Method Around the Horizontal Line: $V(x) = \pi \int_a^b (f(x) - g(x))^2 dx$
- Disc Method Around the Vertical Line: $V(y) = \pi \int_c^d (f(y) - g(y))^2 dy$